§ Notation, etc.

- $G = SL(r)$

$\pi : \tilde{\mathcal{U}}^d_r \longrightarrow \mathcal{U}^d_G$ resolution of singularities

$\tilde{\mathcal{U}}^d_r$: Gieseker sp. — symplectic, $\dim = 2dr$; $\mathcal{U}^d_G$: Uhlenbeck sp. — general G later

(E, φ) or
$$\begin{tikzcd} \mathbb{C}^d \arrow[loop left, "B_1"] \arrow[loop right, "B_2"] \arrow[d, shift right, "J"'] \\ \mathbb{C}^r \arrow[u, shift right, "I"'] \end{tikzcd}$$

- $r = 1$ $\quad \tilde{\mathcal{U}}^d_1 = \mathrm{Hilb}^d \mathbb{C}^2$, $\quad \mathcal{U}^d_G = \mathcal{U}^d_{\{e\}} = S^d \mathbb{A}^2$

- group action

$\mathbb{G} = G \times (\mathbb{C}^*)^2 \curvearrowright \tilde{\mathcal{U}}^d_r, \mathcal{U}^d_G$

$\cup \mathbb{T} = T \times (\mathbb{C}^*)^2$

$H^{[*]}_{\mathbb{G}(,c)}(\tilde{\mathcal{U}}^d_r)$, $H^{[*]}_{\mathbb{T}(,c)}(\tilde{\mathcal{U}}^d_r)$: equivariant cohomology

cpt supports

degree shifted by $2dr$

$[\ ]$: omitted here after

- Heisenberg action

$$\tilde{\mathcal{U}}^d_r \times \tilde{\mathcal{U}}^{d+n}_r \times \mathbb{A}^2 \supset \mathfrak{P}_n := \{(E_1, E_2, x) \mid E_1 \supset E_2,\ \mathrm{Supp}(E_1/E_2) = \{x\}\}$$

$\mathfrak{P}_n$: lagrangian

$$\begin{tikzcd} & \mathfrak{P}_n \arrow[dl] \arrow[dr] & \\ \tilde{\mathcal{U}}^{d+n}_r & & \tilde{\mathcal{U}}^d_r \times \mathbb{A}^2 \end{tikzcd}$$

$\alpha \in H^*_{\mathbb{T},c}(\mathbb{A}^2)$ ($\mathbb{T}$) $\rightsquigarrow$ $P^\Delta_{-n}(\alpha) : H^*_{\mathbb{T}}(\tilde{\mathcal{U}}^d_r) \longrightarrow H^{*+\deg\alpha}_{\mathbb{T}}(\tilde{\mathcal{U}}^{d+n}_r)$ ($\mathbb{T},c$; $\mathbb{T},c$)

creation operator

$P^\Delta_n(\alpha) = $ its adjoint $H^*_{\mathbb{T},c}(\tilde{\mathcal{U}}^{d+n}_r) \to H^{*+\deg\alpha}_{\mathbb{T},c}(\tilde{\mathcal{U}}^d_r)$ ($\mathbb{T}$; $\mathbb{T}$)

inner product : $\underbrace{(-1)^{\dim X/2}}_{\text{sign convention}} \int_X \bullet \cup \bullet$

Fact ($r=1$, N, Grojnowski, r: general Baranovsky)

$$[P_m^{\Delta}(\alpha), P_n^{\Delta}(\beta)] = \langle \alpha, \beta \rangle\, m\, \delta_{m+n,0} \times r$$

(Heisenberg relation)

$r=1$ $\quad \bigoplus_d H^*_{(\mathbb{C}^*)^2}(\mathrm{Hilb}^d)\, \mathfrak{z}^d$ has $\dim_{(\text{over } H^*_{(\mathbb{C}^*)^2}(pt))} = \prod_{d=1}^{\infty} \frac{1}{1-\mathfrak{z}^d}$,

$\therefore$ Heis. rep. is irreducible. $\qquad$ dim Fock

Q. If $r>1$, the representation is not irreducible.
So a larger algebra should act.
What algebra?

Why larger?

Consider $T \curvearrowright \widetilde{\mathcal{U}}_r^d$ $\quad$ fixed pts $= (E, \varphi)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\; \parallel$
$\qquad\qquad\qquad\qquad\qquad\qquad I_1 \oplus \cdots \oplus I_r,\ \varphi_1 \oplus \cdots \oplus \varphi_r$

$$\therefore\ \widetilde{\mathcal{U}}_r^d = \coprod_{d_1+\cdots+d_r=r} \mathrm{Hilb}^{d_i} \mathbb{A}^2$$

$$\therefore\ \bigoplus_d H^*_{\mathbb{T}}(\widetilde{\mathcal{U}}_r^d) \otimes \mathbb{C}(\mathrm{Lie}\,T) \cong \left[\bigoplus_d H^*_{(\mathbb{C}^*)^2}(\mathrm{Hilb}^d) \otimes \mathbb{C}(\mathrm{Lie}\,T)\right]^{\otimes r}$$

quotient field

($H^*_T(pt) = \mathbb{C}[\mathrm{Lie}\,T]$ polynomial ring) $\qquad$ r copy of Fock space!

Therefore the bigger algebra should be $\sim (\text{Heisenberg})^{\otimes r}$.

But it does not act on $\bigoplus H^*_{\mathbb{T}}(\widetilde{\mathcal{U}}_r^d)$.
only after $\otimes$ Frac.

Correct Answer : $W(\mathfrak{gl}_r) = W(\mathfrak{sl}_r) \oplus \mathrm{Heis}.$

above

(Schiffmann-Vasserot

Maulik-Okounkov

§ Stable envelop (Maulik-Okounkov)

∘ situation

$\pi : X \longrightarrow X_0$ — resolution
symplectic, affine variety
- $\mathbb{T}$ - equivariant (linear action)
- $T \subset \mathbb{T}$ preserving symp. form

ex. $X = \widetilde{\mathcal{U}}_r^d \longrightarrow X_0 = \mathcal{U}_G^d$, $\mathbb{T} \supset T$ as above

$X = T^*\mathcal{B} \longrightarrow X_0 = \mathcal{N}$, $\mathbb{T} = T \times \mathbb{C}^* \supset T$

$X \circlearrowleft X^T = \coprod F_\alpha$ $\quad$ F_α : symplectic

dim $2dr$

$$i^* : H_{\mathbb{T}}^{[*]}(X) \longrightarrow H_{\mathbb{T}}^{[* + \mathrm{codim}_{\mathbb{C}} X^T]}(X^T) = \bigoplus_\alpha H_{\mathbb{T}}^{[* + \mathrm{codim} F_\alpha]}(F_\alpha)$$

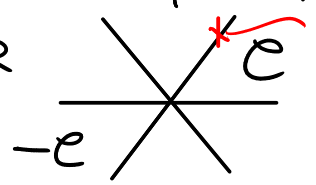
Want

$[*]_X = * + \dim X \to * + \dim X = [*]_{X^T} + \mathrm{codim} X^T$

∘ chamber structure on the space of 1PS's

$\mathrm{Hom}(\mathbb{C}^*, T)$ has "root" hyperplanes $\quad \rho : X^{\rho(\mathbb{C}^*)} \supsetneq X^T$

$\mathrm{Hom}(\mathbb{C}^*, T) \otimes_{\mathbb{Z}} \mathbb{R} = (\mathrm{Lie}\, T)_{\mathbb{R}}$



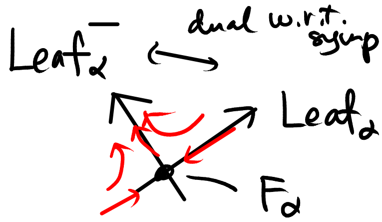
$-\mathfrak{C}$: opposite

Choose a chamber $\mathfrak{C}$ and $\rho : \mathbb{C}^* \to T \in \mathfrak{C}$

Define $X_{(0)} \supset \mathcal{A}_{X_{(0)}} := \{ x \mid \lim_{t \to 0} t \cdot x \text{ exists} \} \longrightarrow X_{(0)}^T$ $\quad$ (t "$= \rho(t)$")

attracting set $\qquad$ depending only on $\mathfrak{C}$

∘ $X \supset \mathrm{Leaf}_\alpha := \{ x \mid \lim_{t \to 0} t \cdot x \in F_\alpha \} \xrightarrow[\text{vectr bdle}]{} F_\alpha$ (BB decomp.)



$$TX|_{F_\alpha} = \bigoplus_{\substack{\text{wt} \\ \text{w.r.t. } \rho}} E(m) \qquad \mathrm{Leaf}^{\pm} = \bigoplus_{m > 0} E(\pm m)$$

$$TF_\alpha = E(0)$$

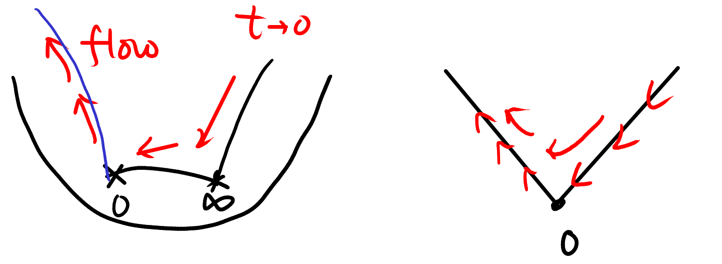
$E(m) \leftrightarrow E(-m)$: dual

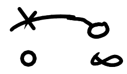
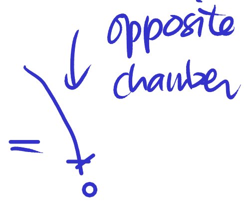
Rem $\quad X \times F_\alpha \supset \mathrm{Leaf}_\alpha \quad$ lagrangian

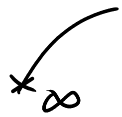
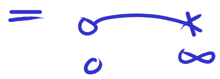
Ex $T^*\mathbb{P}^1 \longrightarrow \mathbb{C}^2/\pm$ $\qquad (T^*\mathbb{P}^1)^T = \{0, \infty\}$

$\circlearrowleft T = \mathbb{C}^* \qquad (z_1, z_2)/\pm 1$

$\qquad\qquad \mapsto (tz_1, t^{-1}z_2)$



$\therefore$ $\mathrm{Leaf}_0 =$  $=$ 

$\mathrm{Leaf}_\infty =$  $=$ 

Rem. Two topologies on $\mathcal{A}_{X_0} = \coprod \mathrm{Leaf}_\alpha$

— disjoint

— $\mathcal{A}_{X_0} \subset X_0$ induced topology

Same for X_0 (affine), different for X

- Define a partial order
$\alpha \geq \beta \iff \overline{\mathrm{Leaf}_\alpha} \cap F_\beta \neq \phi$ $\qquad$ (above example $0 \geq \infty$

- Define $\deg_T$
$T \curvearrowright X^T$ trivial $\quad \therefore \quad H_T^*(X^A) = H^*(X^A) \otimes \underbrace{H_T^*(pt)}_{\cong}$
$\deg_T$ is defined $\longrightarrow \mathbb{C}[\mathrm{Lie}\, T]$
$(\mathrm{Lie}\, T : \deg = 2)$ $\qquad$ polynomial ring

- Steinberg type variety

$$Z_\mathcal{J} := \mathcal{A}_X \times_{X_0^T} X^T \subset X \times X^T \qquad \text{lagrangian subvar.}$$

Consider $\mathcal{A}_X \subset X$ closed subvariety

We use $Z_\mathcal{J}$ as a correspondence between X and X^T, and defines an operator $H^*_{\mathbb{T}}(X^T) \longrightarrow H^*_{\mathbb{T}}(X)$

◦ Irreducible components of $Z_{\mathfrak{g}}$

$$Z_{\mathfrak{g}} = \underbrace{\Delta_X}_{\coprod_\beta \mathrm{Leaf}_\beta} \times_{X_0^T} \underbrace{X^T}_{\coprod_\alpha F_\alpha} = \coprod_{\alpha,\beta} \mathrm{Leaf}_\beta \times_{X_0^T} F_\alpha$$

$$\begin{tikzcd} \coprod_{\alpha,\beta} \mathrm{Leaf}_\beta \times_{X_0^T} F_\alpha \arrow[d, "p"] \\ F_\beta \times_{X_0^T} F_\alpha \end{tikzcd}$$

<u>Lemma</u>. $F_\beta \times_{X_0^T} F_\alpha$: lagrangian

proof)

<u>Fact</u> $F_\beta \longrightarrow \pi_1(F_\beta) \subset X_0^T$, $F_\alpha \longrightarrow \pi_2(F_\alpha) \supset$ semismall

S_ν common stratum

$$F_\beta \times_{X_0^T} F_\alpha = \bigcup_\nu \pi_1^{-1}(S_\nu) \times_{S_\nu} \pi_2^{-1}(S_\nu)$$

dim S_ν + dim fiber of π_1 over S_ν

+ " π_2

$$= \dim S_\nu + \tfrac{1}{2}(\dim F_\beta - \dim S_\nu) + \tfrac{1}{2}(\dim F_\alpha - \dim S_\nu)$$

$$= \tfrac{1}{2}(\dim F_\alpha + \dim F_\beta) \quad //$$

$\{Y_k^T\}$: irr. components of $\coprod_{\alpha,\beta} F_\beta \times_{X_0^T} F_\alpha$

$$Y_k := \overline{p^{-1}(Y_k^T)} \quad \dim = \tfrac{1}{2}(\dim F_\beta + \dim F_\alpha) + \tfrac{1}{2}\operatorname{codim} F_\beta \text{ in } X$$

$$= \tfrac{1}{2}\dim X \times F_\alpha$$

<u>Prop</u> $\{Y_k\}$: irreducible components of $Z_{\mathfrak{g}}$

$$\bigcup_{\beta \leqq \alpha} \overline{\mathrm{Leaf}_\beta \times F_\alpha} \quad X_0^T$$

top !

Th [MO]

$\exists 1\ \mathcal{L} = \mathcal{L}_{\mathfrak{C}} \in H^T_{\cancel{[0]}}(Z_\sigma) \left(= H^{[0]}_T (X \times X^T ; X \times X^T \setminus Z_\sigma) \right)$

s.t. (1) $\mathcal{L}|_{X \times F_\alpha}$ is supported on $\bigcup_{\beta \text{ s.t. } \beta \leqq \alpha} Y_\alpha$

(2) $i_{\alpha,\alpha} : F_\alpha \times F_\alpha \longrightarrow X \times X^T$

$$i_{\alpha,\alpha}^* \mathcal{L} = \pm\, e(\mathrm{Leaf}_\alpha^-) \cap [\Delta_{F_\alpha}]$$

$\uparrow$ vector b'dle over F_α

$*$ normal b'dle of $\mathrm{Leaf}_\alpha^+ = \mathrm{Leaf}_\alpha^-$

(3) $i_{\beta,\alpha} : F_\beta \times F_\alpha \to X \times X^T \quad (\beta < \alpha)$

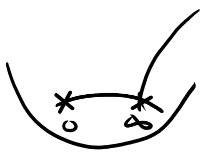
$$\deg_T i_{\beta,\alpha}^* \mathcal{L} < \frac{1}{2} \operatorname{codim} F_\beta$$

$*$ $\operatorname{rk} \mathrm{Leaf}_\beta^\pm = \frac{1}{2} \operatorname{codim} F_\beta$

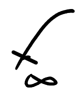
(In fact, in our situation, much stronger condition $i_{\beta,\alpha}^* \mathcal{L} = 0$ is true.)

$\mathcal{L}$ is called the stable envelop.

<u>Ex.</u> $X = T^*\mathbb{P}^1 \curvearrowleft T$



$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\infty$$

$\pm$  $\times \{0\}$ $\pm$  $\times \{\infty\}$

- polarization (sign)

$X \supset F_\alpha \qquad N_{F_\alpha/X} = N^+ \oplus N^-$ dual

$\therefore (-1)^{\mathrm{codim}/2} e(N_{F_\alpha/X}) = e(N^+)^2$

On the other hand, X and F_α are cotangent b'dles

$X \doteqdot T^*M \quad T(T^*M) = TM \oplus T^*M$

$(-1)^{\dim/2} e(TX) = e(TM)^2 \quad \Rightarrow e(TM) = \pm e(N^+)$ in T-equiv. coh.

In many situations ($\widetilde{\mathcal{U}}_r^d$, $T^*\mathcal{B}$), we have a prefered choice of $\sqrt{e(N)}$. We define $\pm$ accordingly.

proof) <u>existence</u>

<u>Fact</u> X, X_0 has a 1 parameter deformation

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \mathfrak{X}_0 \\ p \downarrow & \swarrow & \\ \Delta & & \end{array}$$

s.t. $X_t = p^{-1}(t)$ is affine

& $X_t \xrightarrow{\cong} X_{0,t}$ for $t \neq 0$

T-action extends to $\mathfrak{X}$ (<u>NB.</u> $\mathbb{T}$-action does <u>not</u> extend)

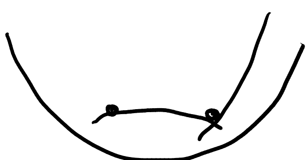
$\widetilde{\mathcal{U}}_r^d \longrightarrow$ deform the moment map equ.

$[B_1, B_2] + IJ = t\,\mathrm{id}$

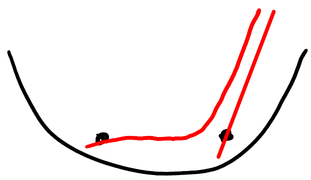
$\mathbb{C}^* \times \mathbb{C}^*$-action does not preserve $\uparrow$

Now $\left(\begin{array}{l} F_\alpha \\ \mathrm{Leaf}_\alpha \end{array} \right.$ also deformed

Leaf_α : closed for $t \neq 0$



$t=0$



$t \neq 0$

$[\mathrm{Leaf}_\alpha]$ at $t \neq 0$

$\mathcal{L}_\alpha := \lim_{t \to 0} [\mathrm{Leaf}_\alpha]$

$i_{\beta,\alpha}^* \, Leaf_\alpha = 0 \quad \Rightarrow \quad i_{\beta,\alpha}^* \mathcal{L}_\alpha = 0$

Uniqueness

$\mathcal{L} = \sum_\alpha \mathcal{L}_\alpha$ according to $X^T = \coprod F_\alpha$

$\mathcal{L}_\alpha = \sum_{k:\, \beta \leq \alpha} a_{Y_k} [Y_k] \qquad a_{Y_k} \in \mathbb{Z}$

(1) : $\beta = \alpha \rightarrow Z = \Delta_{F_\alpha} \subset F_\alpha \times_{X_0^T} F_\alpha$ & $a_Z = \pm 1$

$N_{Leaf_\alpha / X} = Leaf_\alpha^- \Rightarrow i_{\alpha,\alpha}^* \mathcal{L}_\alpha = \pm e(Leaf_\alpha^-) \cap [\Delta_{F_\alpha}]$

Suppose two $\mathcal{L}_\alpha^1, \mathcal{L}_\alpha^2$ $\quad \mathcal{L}_\alpha^1 - \mathcal{L}_\alpha^2 = \sum_{\beta < \alpha} \sum_k a'_{Y_k} [Y_k]$

Take a maximum β_0 among $a'_{Y_k} \neq 0$ for some k

$\Rightarrow$ other terms do not contribute to $i_{\beta_0,\alpha}^*$

$\therefore \quad i_{\beta_0,\alpha}^* (\mathcal{L}_\alpha^1 - \mathcal{L}_\alpha^2) = \sum_{k:\, \beta_0 \alpha} a'_{Y_k} \, e(Leaf_{\beta_0}^-) \cap [Y_k^T]$

invertible in H_T^*

(2) $\Rightarrow a'_{Y_k} = 0 \quad \forall k$ (β_0, α) contradiction !

Proof finish

This $\mathcal{L}$ gives us only a "trivial" thing.
But consider $\mathbb{T} \supset T$

$t \in \mathbb{T}$ commutes with T
$\Rightarrow Z_{\mathfrak{g}}$: invariant under $\mathbb{T}$

$\therefore \quad \mathcal{L} \in H_0^T(Z_{\mathfrak{g}}) = H_0^{\mathbb{T}}(Z_{\mathfrak{g}})$ (as it is spanned by irr. comp.)

$\mathcal{L}_{\mathcal{C}}$ defines an operator

$$\begin{array}{ccc} & \mathcal{L}_{\mathcal{C}} & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathcal{A}_X & & X^T \end{array} \qquad \mathrm{Stab}_{\mathcal{C}} : H^*_{\mathbb{T}}(X^T) \longrightarrow H^{\mathbb{T}}_*(\mathcal{A}_X) = H^*_{\mathbb{T}}(X; X \setminus \mathcal{A}_X)$$

$$\alpha \longmapsto p_{1*}(p_2^* \alpha \cap \mathcal{L}_{\mathcal{C}})$$

(p_1 : proper)

@ transpose $\quad H^*_{T,c}(X) \longrightarrow H^*_{T,c}(X^T)$

is given by $\mathcal{L}$ for $-\mathcal{C}$

changing the role of X & X^T

$$i^* \mathrm{Stab}_{\mathcal{C}} : H^{\mathbb{T}}_*(X^T) = H_*^{\mathbb{T}/T}(\underbrace{X^T}_{= \coprod F_\alpha}) \otimes_{H^*_{\mathbb{T}/T}(pt)} H^*_{\mathbb{T}}(pt)$$

uppertriangular
& diagonal $= e(\mathrm{Leaf}_{\overline{F}_\alpha}) \cup$

$\subset$ top part $= \mathbb{T}$ wts $\neq 0$

<u>NB</u> $\quad i^*_{\beta\alpha} \mathcal{L} \neq 0$ on $H^*_{\mathbb{T}}$

$\therefore$ $\mathrm{Stab}_{\mathcal{C}}$ becomes invertible after $\otimes \underbrace{\mathrm{Frac}\, H^*_{\mathbb{T}}(pt)}_{= \mathbb{Q}(\mathrm{Lie}\,\mathbb{T})}$

§ R-matrix and Yangian

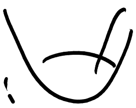
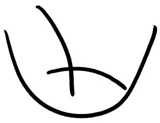
(So Stab $= \sqrt{\text{R-matrix}}$

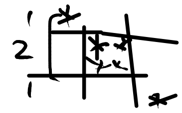
Def. $R_{\mathfrak{C}',\mathfrak{C}} = \mathrm{Stab}_{\mathfrak{C}'}^{-1} \circ \mathrm{Stab}_{\mathfrak{C}}$

R-matrix $\in \mathrm{End}(H^*_{\mathbb{T}}(X^T) \otimes \mathbb{Q}(\mathrm{Lie}\,\mathbb{T}))$

$$H^*_{\mathbb{T}}(X^T) \xrightarrow[\cong]{\mathrm{Stab}_{\mathfrak{C}}} H^{\mathbb{T}}_*(\mathcal{A}^{\mathfrak{C}}_X) \to H^{\mathbb{T}}_*(X) \leftarrow H^{\mathbb{T}}_*(\mathcal{A}^{\mathfrak{C}'}_X) \leftarrow H^*_{\mathbb{T}}(X^T)$$

$R_{\mathfrak{C}',\mathfrak{C}}$

Ex. $X = T^*\mathbb{P}^1$ $\quad \mathfrak{C}:$  $\quad \mathfrak{C}^{op}$  $\quad X^T = \{0, \infty\}$

$R_{\mathfrak{C}^{op},\mathfrak{C}}$ = block of Yang's R-matrix 

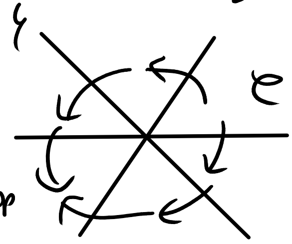
$(Y(\mathfrak{sl}_2)$ $\mathbb{C}^2 \otimes \mathbb{C}^2$ $\quad R = 1 - \frac{\hbar P}{u}$ $\quad P = \sum_{i,j=1}^{2} e_{ij} \otimes e_{ji}$

up to normalization $\quad \mathbb{C}[\hbar] = H^*_{\mathbb{C}^*}(pt)$

○ Yang-Baxter equation

Suppose T : 2 dim $\quad \mathrm{Lie}\,T = \{a_1 + a_2 + a_3 = 0\}$

$\begin{pmatrix} \cap \\ SL_3 \end{pmatrix}$ $\quad$ Suppose chambers



$a_2 = a_3$

$\mathfrak{C}$

$a_1 = a_2$

$\mathfrak{C}^{op}$

$a_1 = a_3$

$$R_{12} R_{13} R_{23} \;\; \| \;\; R_{23} R_{13} R_{12}$$

depending only on $\begin{pmatrix} [\,*\,] & 0 \\ 0 & * \end{pmatrix} \subset T$

eg. $H^*_{\mathbb{T}}(\widetilde{\mathcal{U}}^d_r)$ $\quad r = 3$ $\quad SL_3 \supset T = \mathbb{C}^* \times \mathbb{C}^* > \mathbb{C}^* = \begin{array}{c} 2 \\ 1 \end{array}\left[\begin{array}{cc|c} * & & 0 \\ & * & \\ \hline 0 & & * \end{array}\right]$

$\det = 1$

$$(\widetilde{\mathcal{U}}^d_{r=3})^{\mathbb{C}^*} = \{(E_1, \varphi_1) \oplus (E_2, \varphi_2)\}$$

$$= \coprod_{d_1 + d_2 = d} \widetilde{\mathcal{U}}^{d_1}_{r=2} \times \underbrace{\widetilde{\mathcal{U}}^{d_2}_{r=1}}_{= \mathrm{Hilb}^{d_2}}$$

$\Rightarrow$ above chamber structure, and hence YB equ.!

☆ Baranovsky's Heisenberg operators preserves $H_*^T(\mathcal{J})$

Under
$$H_*^T(\mathcal{J}) \cong \bigoplus_d H_*^T((\widetilde{\mathcal{U}}_r^d)^T)$$
$$= \Big\{ \bigoplus_d H_*^T(\mathrm{Hilb}^d) \Big\}^{\otimes r}$$

<u>Prop</u>. It is given by the diagonal Heisenberg, i.e.,
$$P_n^{\triangle}(\alpha) = \sum_{i=1}^{r} \mathrm{id} \otimes \cdots \otimes \underbrace{P_n(\alpha)}_{i\text{th factor}} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}$$

☆ R-matrix intertwines $P_n^{\triangle}(\alpha)$ by definition

$$\begin{tikzcd} H^*_T(X^T) \arrow[r, "\mathrm{Stab}_C", "\cong"'] \arrow[rrrr, bend right, "R_{C',C}"'] & H_*^T(\alpha_X^C) \arrow[r] & H_*^T(X) & H_*^T(\alpha_X^{C'}) \arrow[l] & H^*_T(X^T) \arrow[l] \end{tikzcd}$$

commute by definition

<u>Rem</u>. $r = 1$
$$\underset{\substack{\| \\ P_n \otimes 1}}{P_n^{(1)}} + \underset{\substack{\| \\ 1 \otimes P_n}}{P_n^{(2)}} \text{ , but not } P_n^{(1)} - P_n^{(2)}$$

$c_1(\underset{\uparrow}{E})$ commutes with R

tautological bdle

$\rightsquigarrow$ Virasoro op

※ R-matrix $\rightsquigarrow$ Yangian   RTT construction
Fadeev – Reshetikhin – Takhadzhyan

- $T^*\mathbb{P}^1$ (more generally T^*Gr = quiver variety of type A) $\rightsquigarrow Y(\mathfrak{sl}_2)$ recover   N, Varagnolo. $U_q(\mathfrak{g})$
- $\widetilde{\mathcal{U}}^d_r \rightsquigarrow Y(\widehat{\mathfrak{gl}}_1)$ " Heis.   new!

<u>Data</u> (a simpler version)

① F : vector space / $\mathbb{K}$: ring $\supset \mathbb{Q}$

e.g. $F = \bigoplus_d H^*_{(\mathbb{C}^*)^2}(\mathrm{Hilb}^d)$, $\mathbb{K} = H^*_{(\mathbb{C}^*)^2}(pt)$

② $R(u) \in \mathrm{End}(F\otimes F)(u)$; a matrix-valued rational fct in u

s.t. YBE is satisfied

normalization $R(\infty) = 1$

$R(u) - 1$ divisible by $\hbar \in \mathbb{K}$ $\leftarrow (\mathbb{C}^*)^2/\mathbb{C}^*_{hyp.}$

<u>Construction</u>

$W \equiv W_p := F[u_1] \otimes \cdots \otimes F[u_p]$

$R_{F,W} := R(u-u_p) \cdots R(u-u_1)$ $F \otimes F[u_1] \otimes \cdots \otimes F[u_p]$

$\in \mathrm{End}(F\otimes W) = \mathrm{End}F \otimes \mathrm{End}W$

Define $Y \subset \prod_p \mathrm{End}_{\mathbb{K}[u_1,\dots,u_p]} W$ as an algebra generated by coefficients of the following elements $\times\left(\frac{1}{\hbar}\right)$

– Choose a base of F (possibly ∞-dim'l)

$t_{ij}(u) := \big((i,j)\text{-matrix entry of } R_{F,W}\big) \in \prod_p \mathrm{End}_{\mathbb{K}[u_1,\dots,u_p]} W[[u^{-1}]]$

$$\frac{1}{u-u_i} = 1 + \frac{u_i}{u} + \frac{u_i^2}{u^2} + \cdots \quad \to \text{polynomial in } u_i$$

$T(u) := (t_{ij}(u)) \in \mathrm{End}(F) \otimes Y[[u^{-1}]]$

YB equ. $\Longrightarrow$ T satisfies the RTT relation

$$T_2(u_2)T_1(u_1)R(u_1-u_2) = R(u_1-u_2)T_1(u_1)T_2(u_2)$$

- Yang's R-matrix $\Longrightarrow$ $Y(\mathfrak{gl}_2)$
- Hilb $=: Y(\hat{\mathfrak{gl}}_1)$ (definition)

∘ What is R concretely?

$r=2$

$G=SL_2$ $\quad \tilde{\mathcal{U}}^d_2 \supset (\tilde{\mathcal{U}}^d_2)^{\mathbb{C}^*} = \coprod_{d=d_1+d_2} \mathrm{Hilb}^{d_1}\times \mathrm{Hilb}^{d_2}$

$\mathfrak{h}=\{(a_1,a_2) \mid a_1+a_2=0\}$ $\quad \therefore R \circlearrowright \left(\bigoplus H^*_T(\mathrm{Hilb}^{d_1})\otimes \mathbb{Q}(\mathrm{Lie}\,T)\right)^{\otimes 2} \circlearrowleft$ irreducible

$\mathrm{Fock}^{\otimes 2}$ $\quad P_n^{(1)}, P_n^{(2)}$

commute

$P_n^{\Delta} = P_n^{(1)} + P_n^{(2)}$

$P_n^{-} = P_n^{(1)} - P_n^{(2)}$

$\therefore$ R is written by P_n^-

<u>Lemma</u> R is determined by its matrix elements on vac $\otimes$ Fock $\leadsto$ $\mathrm{Hilb}^0\times\mathrm{Hilb}^d$

Feigin-Fuchs Heis. $\to$ Virasoro

- $P_n^- := P_n^-(1)$ $\left(\rightsquigarrow \text{ comm } \langle 1,1\rangle_{\mathbb{C}^2} = \frac{1}{\varepsilon_1\varepsilon_2}\right)$

- Put $P_0^- := \frac{1}{\varepsilon_1\varepsilon_2}(a^1 - a^2 - (\varepsilon_1+\varepsilon_2))$ as convention

$$L_n := -\frac{1}{4}\sum_m :P_m^- P_{n-m}^-: - \frac{n+1}{2}\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2} P_n^- \qquad H^0_{(\mathbb{C}^*)^2}(pt)$$

$\Longrightarrow$ L_n satisfies the Virasoro relation

$$[L_m, L_n] = (m-n)L_{m+n} + \left(1 + \frac{6(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}\right)\delta_{m,-n}\frac{m^3-m}{12}$$

$$L_0|\mathrm{vac}\rangle = -\frac{1}{4}\left(\frac{(a_1-a_2)^2}{\varepsilon_1\varepsilon_2} - \frac{(\varepsilon_1+\varepsilon_2)^2}{\varepsilon_1\varepsilon_2}\right)|\mathrm{vac}\rangle$$

highest wt

$a_1 \leftrightarrow a_2$ invariant $\therefore$ $\mathrm{Fock}^- \circlearrowleft \exists 1$ reflection op.

Virasoro intertwiner s.t. $|\mathrm{vac}\rangle \mapsto |\mathrm{vac}\rangle$

<u>Th</u> [MO] $R =$ reflection operator